Assignment 11.

Scwarz Lemma. Laurent Series.

This assignment is due Wednesday, April 17 (counting extra extension, but not counting regular one-week extension. If you choose to take regular one-week extension, that will make this HW due Wednesday, April 24.). Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

Please note that Problem 2 is excluded from the denominator of the grade.

1. Schwarz Lemma

Reminder. **Schwarz Lemma.** Let $f(z)$ be a function analytic on the disc $K : |z| < R$, $f(0) = 0$ and suppose that $|f(z)| \le M < \infty$ for all $z \in K$. Then $|f(z)| \le \frac{M}{R}|z|$ for all $z \in K$. Moreover, $|f'(0)| \le \frac{M}{R}$. Either equality is achieved if and only if f is a linear function $f(z) = e^{i\alpha}\frac{M}{R}z$, where $\alpha \in \mathbb{R}$.

(1) (a) Find a Möbius transformation g that sends the disc $K : |z| < R$ bijectively to itself, and sends 0 to $a \in K$.

(b) Generalize Schwarz Lemma to the case $f(a) = 0$ ($a \in \mathbb{C},\ |a| < R$). (*Hint:* Consider $f(g(z))$, where a Möbius transformation g is picked so that Scwarz Lemma applies to $f(g)$.)

(2) Let f be a function $f : \overline{H} \to \overline{H}$, where H is the right half plane $H = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$. Suppose f is bijective, analytic on H, continuous on $\overline{H}$, and $f(1) \in \mathbb{R}$. Prove that for every $z \in H$,

$$\left|\frac{f(z) - f(1)}{f(z) + f(1)}\right| \le \frac{z-1}{z+1}.$$

(*Hint:* Show that $w(z) = \frac{z-1}{z+1}$ is a bijection between H and unit disc. Apply Schwarz Lemma to an appropriate function.)

Comment. By using same sort of tricks (composing and precomposing with appropriate Möbius transformations), one can prove (ant it is a nice exercise to do so) the following *Schwarz-Pick* theorem:
If f is a bijective map from the closed unit disc $\overline{D}$ to itself, analytic on D, and continuous on $\overline{D}$, then for all $z_1, z_2, z \in D$

$$\left|\frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)}f(z_2)}\right| \le \left|\frac{z_1 - z_2}{1 - \overline{z_1}z_2}\right| \quad \text{and} \quad \frac{|f'(z)|}{1 - |f(z)|^2} \le \frac{1}{1 - |z|^2}.$$

(By the way: is $|\frac{z_1 - z_2}{1-\overline{z_1}z_2}|$ a metric? If it is, how can one interpret the first inequality?)

— see next page —

2. Laurent Series

(3) Expand the function

$$f(z) = \frac{1}{(z-a)} \quad (a \in \mathbb{C},\ a \neq 0)$$

in a Laurent series in the annuli

(a) $0 < |z| < |a|$,

(b) $|a| < |z|$.

(4) Expand the function

$$f(z) = \frac{1}{(z-a)(z-b)} \quad (a, b \in \mathbb{C},\ 0 < |a| < |b|)$$

in a Laurent series in the annuli

(a) $0 < |z| < |a|$,

(b) $|a| < |z| < |b|$,

(c) $|b| < |z|$.

(*Hint:* Decompose into elementary fractions and use the previous problem.)

(5) Expand the function

$$f(z) = \frac{1}{(z-a)^k}, \quad (a \in \mathbb{C},\ a \neq 0, \quad k \in \mathbb{Z},\ k > 0)$$

in a Laurent series in the annuli

(a) $0 < |z| < |a|$,

(b) $|a| < |z|$.

(*Hint:* Use Weierstrass theorem).